PROOF OF THE RIEMANN MAPPING THEOREM

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Abstract. Unfortunately, the proof of the Riemann Mapping Theorem in the wonderful book [1] is slightly incorrect, since Schwarz' Lemma requires 0 to be fixed by the map this lemma is applied to. In this note we correct this inaccuracy. For the convenience of the reader, we provide a complete proof, which is identical to Rudin's in all steps but the beginning of Step 2. The references are relative to [1].

Theorem 1 (Riemann Mapping Theorem, Thm. 14.8)**.** *Every simply connected domain Ω in $\mathbb{C}$ (other than $\mathbb{C}$ itself) is conformally equivalent to the open unit disc $\mathbb{D}$.*

Proof. Let Ω be a simply connected domain in $\mathbb{C}$, $\Omega \neq \mathbb{C}$. Recall that then for each holomorphic function $f\colon \Omega \to \mathbb{C} \setminus \{0\}$ there is a holomorphic function $g\colon \Omega \to \mathbb{C}$ such that

$$f(z) = e^{g(z)}$$

(a complex logarithm). Thus, $h\colon \Omega \to \mathbb{C}$,

$$h(z) := e^{\frac{1}{2}g(z)}$$

is a holomorphic function satisfying $h^2 = f$ (a holomorphic square root of f).

Let Σ be the set of injective holomorphic maps $\psi\colon \Omega \to \mathbb{C}$ such that $\psi(\Omega) \subseteq \mathbb{D}$. If $\psi(\Omega) = \mathbb{D}$ for some $\psi \in \Sigma$, then ψ is a biholomorphic map from Ω onto $\mathbb{D}$ by Thm. 10.33. We will show that Σ contains such a ψ.

Step 1: We claim that Σ is non-empty.

Let $w_0 \in \mathbb{C} \setminus \Omega$. Then there exists a holomorphic map $\varphi\colon \Omega \to \mathbb{C}$ such that $\varphi(z)^2 = z - w_0$.

Let $z_1, z_2 \in \Omega$ such that $\varphi(z_1) = \varphi(z_2)$. Then

$$z_1 - w_0 = \varphi(z_1)^2 = \varphi(z_2)^2 = z_2 - w_0,$$

hence $z_1 = z_2$. Therefore φ is injective. Analogously, one sees that

$$\varphi(z_1) \neq -\varphi(z_2) \tag{1}$$

for each pair $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$.

By the Open Mapping Theorem φ is open. Thus, $\varphi(\Omega)$ contains a disc $U_r(a)$ with $0 < r < |a|$ (for some a). Then (1) implies

$$U_1(-a) \cap \varphi(\Omega) = \emptyset.$$

Since $\varphi(\Omega)$ is open, it follows that

$$\overline{U_r(-a)} \cap \varphi(\Omega) = \emptyset.$$

Hence $|\varphi(z)+a| > r$ for all $z \in \Omega$. Therefore the map

$$\psi := \frac{r}{\varphi + a}$$

is in Σ.

Step 2: Let $\psi \in \Sigma$ such that $\psi(\Omega) \neq \mathbb{D}$. Further let $z_0 \in \Omega$. We claim that there is $\psi_1 \in \Sigma$ such that

$$|\psi_1'(z_0)| > |\psi'(z_0)|.$$

Set $w := \psi(z_0)$ and consider the map

$$\chi := \varphi_w \circ \psi$$

where φ_w is given by Def. 12.3. Since $\varphi_w \circ \psi(\Omega) \subseteq \mathbb{D}$ and $\varphi_w \circ \psi$ is injective, we see that $\chi \in \Sigma$. Further, since φ_w is biholomorphic on $\mathbb{D}$, we find $\chi(\Omega) \subsetneq \mathbb{D}$.

Let $\alpha \in \mathbb{D} \setminus \chi(\Omega)$. Then $\varphi_\alpha \circ \chi \in \Sigma$. Since the unique root α of φ_α is not in $\chi(\Omega)$, the map $\varphi_\alpha \circ \chi$ does not have a root.

Hence there is a holomorphic function $g \colon \Omega \to \mathbb{C}$ such that $g^2 = \varphi_\alpha \circ \chi$. Since g^2 is injective, g is so. Moreover we have

$$|g(z)|^2 = |\varphi_\alpha(\chi(z))| < 1$$

for all $z \in \Omega$. Thus $g \in \Sigma$.

Let $\beta := g(z_0)$. Then

$$\psi_1 := \varphi_\beta \circ g \in \Sigma$$

and $\psi_1(z_0) = 0$. With $s(z) := z^2$ it follows that

$$\chi = \varphi_{-\alpha} \circ s \circ g = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \circ \psi_1.$$

Set

$$F := \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}.$$

Since $\psi_1(z_0) = 0$, the chain rule implies

$$\chi'(z_0) = F'(0)\psi_1'(z_0). \tag{2}$$

Further, $F(\mathbb{D}) \subseteq \mathbb{D}$,

$$F(0) = \chi(z_0) = \varphi_w(\psi(z_0)) = 0,$$

and F is not injective (otherwise s would be injective). Hence F is not a rotation. Therefore Schwarz' Lemma shows

$$|F'(0)| < 1.$$

Taking absolute values in (2) we find

$$|\chi'(z_0)| < |\psi_1'(z_0)|.$$

Using Thm. 12.4 we get

$$\chi'(z_0) = \varphi_w'(w)\psi'(z_0) = \frac{\psi'(z_0)}{1-|w|^2}.$$

Sin ce $0 < 1 - |w|^2 < 1$, it follows that $|\psi'(z_0)| < |\chi'(z_0)|$, and thus

$$|\psi'(z_0)| < |\psi_1'(z_0)|.$$

We fix $z_0 \in \Omega$ and define

$$\lambda := \sup\{|\psi'(z_0)| \mid \psi \in \Sigma\}.$$

Step 3: We claim that there is $\vartheta \in \Sigma$ such that $|\vartheta'(z_0)| = \lambda$. Step 2 then shows $\vartheta(\Omega) = \mathbb{D}$, which completes the proof.

Since $|\psi(z)| < 1$ for all $\psi \in \Sigma$ and $z \in \Omega$, the set Σ is locally bounded. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence in Σ such that $(|\psi_n'(z_0)|)_{n\in\mathbb{N}}$ converges to λ. By the Theorem of Montel there is a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ which converges uniformly on compact sets to a holomorphic function $\vartheta\colon \Omega \to \mathbb{C}$.

The Convergence Theorem of Weierstrass shows that $(\psi'_{n_k})_{k\in\mathbb{N}}$ converges to $\vartheta'(z_0)$. Thus

$$|\vartheta'(z_0)| = \lambda \quad > 0.$$

Since $\psi_n(\Omega) \subseteq \mathbb{D}$ and $\psi_{n_k} \to \vartheta$ pointwise, $\vartheta(\Omega) \subseteq \overline{\mathbb{D}}$. The Maximum Principle shows $\vartheta(\Omega) \subseteq \mathbb{D}$. It remains to show that ϑ is injective.

Let $z_1, z_2 \in \Omega$, $z_1 \neq z_2$. Set $\alpha := \vartheta(z_1)$ and $\alpha_k := \psi_{n_k}(z_1)$ for $k \in \mathbb{N}$. By the Identity Theorem there is $r > 0$ such that $\vartheta - \alpha$ does not have roots on $D := \overline{U_r(z_2)} \subseteq \Omega$ other than possibly z_2. After shrinking r we may assume $z_1 \notin D$. Let

$$\varepsilon := \min\{|\vartheta(z) - \alpha| \mid z \in \partial D\} > 0.$$

Since $(\psi_{n_k} - \alpha_k)_{k\in\mathbb{N}}$ converges uniformly to $\vartheta - \alpha$ on D, there is $l \in \mathbb{N}$ such that

$$\|(\vartheta - \alpha) - (\psi_{n_l} - \alpha_l)\|_\infty < \varepsilon$$

on D. The Theorem of Rouché shows that on D the map $\vartheta - \alpha$ has as many roots as $\psi_{n_l} - \alpha_l$. Since $\psi_{n_l} - \alpha_l$ is injective on Ω and the root z_1 of $\psi_{n_l} - \alpha_l$ is not contained in D, the map $\psi_{n_l} - \alpha_l$ does not have any root in D. Therefore $\vartheta - \alpha$ does not have any root in D. Thus, $\vartheta(z_2) \neq \alpha = \vartheta(z_1)$. □

References

[1] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

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